# Integration:
## Integration using Substitution

## When to use Integration by Substitution

Integration by Substitution is the first technique we try when the integral is not basic enough to be evaluated using one of the anti-derivatives that are given in the standard tables or we can not directly see what the integral will be. The idea is to define a new variable which will allow the difficult starting integrand to be changed from the old variable to a new integrand which is in terms of the new variable.

For example if our initial integral is $\left(\int f(x)\,dx\right)$, we can define a new variable $u = u(x)$ which will turn our starting integral into a new integral of the form $\left(\int g(u)\,du\right)$. It is important here to note that once we have changed variable there must not be any functions of the old variable $(x,\ dx)$ remaining in the new integral $(u,\ du)$.

It is very likely that you have used Integration by Substitution before on relatively simple integrals (such as the basic example below) without realising that the framework can also be used for the more difficult examples presented below.

## Choosing our new variable $u(x)$

The choice for $u(x)$ is critical in Integration by Substitution as we need to substitute all terms involving the old variables before we can evaluate the new integral in terms of the new variables.
A basic rule of thumb is that when we choose our substitution variable, the substitution will be useful if the rest of the non substituted integral expression is proportional to the derivative of the substitution. This is best explained through examples as shown below.

## Basic Example: $\displaystyle\int \cos(2x+7)\,dx$

Using our substitution criteria above for $u(x)$ we see that that $u(x) = (2x+7)$ will be our choice for the substitution variable as:

$$u(x) = (2x+7) \qquad \Longrightarrow \qquad \frac{du}{dx} = 2 \qquad \Longrightarrow \qquad \frac{du}{2} = dx$$

Now substituting this along with $u(x) = (2x+7)$ into the the original integral (in terms of $x$ and $dx$) allows us to change to a new integral in terms of $u$ and $du$.

$$\int \cos(2x+7)\,dx \;=\; \int \cos(u)\left(\frac{du}{2}\right) \;=\; \frac{1}{2}\int \cos(u)\,du$$

We can now evaluate this using the usual basic integral techniques and substituting back in $u(x)$, where $u(x)$ is our original substitution $u(x) = (2x+7)$ we get:

$$\frac{1}{2}\int \cos(u)\,du \;=\; \frac{1}{2}(\sin(u)) + C \;=\; \frac{1}{2}\sin(u) + C$$

$$\boxed{\int \cos(2x+7)\,dx = \frac{1}{2}\sin(2x+7) + C}$$

Try checking that this answer is correct by differentiating the final answer $\dfrac{1}{2}\sin(2x+7) + C$.

## Difficult Example: $\displaystyle\int \frac{3}{x(\ln(x))^2}\,dx$

Using our substitution criteria above for $u(x)$ we see that that $u(x) = \ln(x)$ will be our choice for the substitution variable as:

$$u(x) = \ln(x) \qquad \Longrightarrow \qquad \frac{du}{dx} = \frac{1}{x} \qquad \Longrightarrow \qquad du = \frac{dx}{x}$$

Now substituting this along with $u(x) = \ln(x)$ into the the original integral (in terms of $x$ and $dx$) allows us to change to a new integral in terms of $u$ and $du$.

$$\int \frac{3}{x(\ln(x))^2}\,dx \;=\; 3\int \frac{1}{(\ln(x))^2}\frac{(dx)}{x} \qquad \Longrightarrow \qquad 3\int \frac{1}{(u)^2}\,(du) \;=\; 3\int \frac{1}{u^2}\,du$$

We can now evaluate this using the usual basic integral (integration) techniques:

$$3\int \frac{1}{u^2}\,du \;=\; 3\int u^{-2}\,du \;=\; 3\left(\frac{u^{-1}}{-1}\right) + C \;=\; -\frac{3}{u} + C$$

We can now move back to the starting variable $x$ by substituting back in $u(x)$, where $u(x)$ is our original substitution $u(x) = \ln(x)$. Hence the final solution is given by:

$$\int \frac{3}{x(\ln(x))^2}\,dx \;=\; 3\int \frac{1}{u^2}\,du \;=\; -\frac{3}{u} + C \;=\; -\frac{3}{\ln(x)} + C$$

$$\boxed{\int \frac{3}{x(\ln(x))^2}\,dx \;=\; -\frac{3}{\ln(x)} + C}$$

Try checking that this answer is correct by differentiating the final answer $-\dfrac{3}{\ln(x)} + C$

## Definite Integral Example with Limits: $\displaystyle\int_0^1 \frac{x}{(3x^2+5)^4}\,dx$

Using our substitution criteria above for $u(x)$ we see that that $u(x) = (3x^2 + 5)$ will be our choice for the substitution variable as:

$$u(x) = (3x^2 + 5) \qquad \Longrightarrow \qquad \frac{du}{dx} = 6x \qquad \Longrightarrow \qquad \frac{du}{6} = x\,dx$$

Now substituting this along with $u(x) = (3x^2+5)$ into the the original integral (in terms of $x$ and $dx$) allows us to change to a new integral in terms of $u$ and $du$. As we have a definite integral we must also change the limits to the new substitution variable $u(x)$. $x = 0$ gives $u(x = 0) = (3(0)^2 + 5) = 5$ and $x = 1$ gives $u(x = 1) = (3(1)^2 + 5) = 8$. This gives:

$$\int_0^1 \frac{x}{(3x^2+5)^4}\,dx \;=\; \int_0^1 \frac{1}{(3x^2+5)^4}\frac{(x\,dx)}{1} \qquad \Longrightarrow \qquad \int_5^8 \frac{1}{(u)^4}\left(\frac{du}{6}\right) \;=\; \frac{1}{6}\int_5^8 \frac{1}{u^4}\,du$$

We can now evaluate this using the usual basic integral (integration) techniques:

$$\frac{1}{6}\int_5^8 \frac{1}{u^4}\,du \;=\; \frac{1}{6}\int_5^8 u^{-4}\,du \;=\; \frac{1}{6}\left(\frac{u^{-3}}{-3}\right)\Big|_5^8 \;=\; -\frac{1}{18u^3}\Big|_5^8$$

$$-\frac{1}{18u^3}\Big|_5^8 \;=\; \left(-\frac{1}{18(8)^3}\right) - \left(-\frac{1}{18(5)^3}\right) \;=\; \frac{43}{128000} \approx 0.00033594$$